1. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ with period $T > 0$ such that, on the interval $[0, T)$, it takes the form

$$f(x) = \begin{cases} +1, & \frac{T}{2} \leqslant x < T, \\ -1, & 0 \leqslant x < \frac{T}{2}. \end{cases}$$

Compute the Fourier series of f.

2. Let us consider an integral equation of the following form:

$$u(t) = g(t) + \int_0^t k(t-s)u(s)\,ds.$$

In the above, $k, g : [0, +\infty) \to \mathbb{R}$ are given piecewise continuous functions and we are solving for a function $u : [0, +\infty) \to \mathbb{R}$. *Remark:* The above equation is a special case of a Volterra equation of second kind. These equations arise naturally in models dynamic systems where past values of a variable influence the current value with a weight determined by the *kernel* function k.

(a) Assuming that both g and k are such so that their Laplace transform is well-defined in some half-space of the form $\{z : \mathrm{Re}(z) > a\}$, find an expression for the Laplace transform of u.

(b) Find u in the case when $g(t) = t$ and $k(t) = e^{-t}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be an *odd*, L-periodic function. Using Fourier series, find an odd and L-periodic solution u of the biharmonic equation

$$\frac{d^4u}{dx^4} = f.$$

4. Consider the following initial state on the interval $I = [0, 2L]$:

$$u_0(x) = \begin{cases} x, & 0 \leqslant x \leqslant L, \\ 2L - x, & L \leqslant x \leqslant 2L. \end{cases}$$

Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

with initial conditions

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = 0$$

and Dirichlet boundary conditions at $x = 0, 2L$:

$$u(0, t) = 0, \quad u(2L, t) = 0.$$

Hint: First extend u, u_0 as odd periodic functions in the variable $x \in \mathbb{R}$; what should be the the period for this extension?

5. For $\kappa > 0$, let us consider the heat equation

$$\frac{\partial u}{\partial t}(x,t) = \kappa \frac{\partial^2 u}{\partial x^2}(x,t), \quad t > 0 \, x \in \mathbb{R}. \tag{1}$$

(a) Show that, for any solution u with $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm + \infty$ and any $t_2 \geqslant t_1$, we have

$$\int_{-\infty}^{+\infty} u(x,t_1)\, dx = \int_{-\infty}^{+\infty} u(x,t_2)\, dx.$$

(Hint: Compute the derivative $\partial_t \int_{-\infty}^{+\infty} u(x,t)\, dx$*.)*

(b) Compute the solution of (1) with initial data

$$u(x,0) = \frac{1}{\sqrt{4\pi\tau\kappa}} e^{-\frac{x^2}{4\tau\kappa}}$$

for some given $\tau > 0$. Deduce, in particular, that the heat evolution of a Gaussian function is a Gaussian function at any fixed time. *(Hint: You will need to recall what is the Fourier transform of a Gaussian function, see Ex. 8.3)*

6. So far, we have only considered cases of *homogeneous* boundary conditions (namely boundary conditions which are invariant if we replace the unknown function $u(x,t)$ with $\lambda \cdot u(x,t)$; for example, Dirichlet conditions $u(x_0,t) = 0$ or Neumann conditions $\partial_x u(x_0,t) = 0$). Let us now consider the question of how to handle inhomogeneous boundary conditions.

To this end, let us consider the following inhomogeneous initial-boundary value problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) & \text{for } x \in (0,1),\, t > 0, \\ u(x,0) = u_0(x), \\ u(0,t) = g_0(t), \quad u(1,t) = g_1(t), & \text{for } t > 0, \end{cases}$$

where $f : (0,1) \times (0,+\infty) \to \mathbb{R}$, $u_0 : (0,1) \to \mathbb{R}$ and $g_0, g_1 : [0,+\infty) \to \mathbb{R}$ are continuous functions.

Defining

$$a(x,t) = g_0(t) \cdot (1-x) + g_1(t) \cdot x,$$

show that, if

$$w(x,t) \doteq u(x,t) - a(x,t),$$

then w solves a heat equation with source term $f(x,t) - \frac{\partial a}{\partial t}(x,t) + \frac{\partial^2 a}{\partial x^2}(x,t)$ and *homogeneous* (in fact, Dirichlet) boundary conditions at $x = 0, 1$.